Quiz 10

November 19th, 2026 15 minutes | 5 points

Name _____

Instructions. Answer all four parts. Show your work and briefly justify each answer.

1. Can classical MDS stretch distances? (Homework 09, Problem 5) 5 points

Let $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^d$, with mean $\bar{\mathbf{x}}$, and let $\mathbf{X}$ have rows $(\mathbf{x}_i - \bar{\mathbf{x}})^\top$. The input distances for MDS are $\delta_{ij} = \|\mathbf{x}_i - \mathbf{x}_j\|_2$. Write the thin SVD of $\mathbf{X}$ as $\mathbf{X} = \mathbf{U}_r \Sigma_r \mathbf{V}_r^\top$, with singular values $\sigma_1 \geq \cdots \geq \sigma_r > 0$. For $1 \leq m \leq r$, let $\mathbf{U}_m$ and $\mathbf{V}_m$ contain the first m left and right singular vectors, respectively, and let $\Sigma_m = \operatorname{diag}(\sigma_1, \dots, \sigma_m)$.

You may use that classical MDS on Euclidean distances returns the rows $\mathbf{y}_i^\top$ of

$$\mathbf{Y} = \mathbf{U}_m \Sigma_m = \mathbf{X}\mathbf{V}_m,$$

up to rotation or reflection.

Your goal is to show that MDS cannot increase any pairwise distance,

$$\|\mathbf{y}_i - \mathbf{y}_j\|_2 \leq \|\mathbf{x}_i - \mathbf{x}_j\|_2,$$

and to determine when equality holds.

Fix a pair i, j and write $\mathbf{q} = \mathbf{x}_i - \mathbf{x}_j$. Let $\mathbf{P} = \mathbf{V}_m \mathbf{V}_m^\top$, the orthogonal projection onto $S = \operatorname{span}\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$.

(a) Using $\mathbf{Y} = \mathbf{X}\mathbf{V}_m$, show that $\mathbf{y}_i - \mathbf{y}_j = \mathbf{V}_m^\top \mathbf{q}$. (1 point)

Solution. Row i of $\mathbf{Y} = \mathbf{X}\mathbf{V}_m$ gives $\mathbf{y}_i^\top = (\mathbf{x}_i - \bar{\mathbf{x}})^\top \mathbf{V}_m$. Transposing and subtracting the corresponding expression for j gives

$$\mathbf{y}_i - \mathbf{y}_j = \mathbf{V}_m^\top ((\mathbf{x}_i - \bar{\mathbf{x}}) - (\mathbf{x}_j - \bar{\mathbf{x}})) = \mathbf{V}_m^\top \mathbf{q}.$$

(b) Show that $\|\mathbf{V}_m^\top \mathbf{q}\|_2^2 = \|\mathbf{P}\mathbf{q}\|_2^2$. (1 point)

Solution. Since $\mathbf{P}$ is an orthogonal projection, $\mathbf{P}^\top \mathbf{P} = \mathbf{P}$. Hence

$$\|\mathbf{V}_m^\top \mathbf{q}\|_2^2 = \mathbf{q}^\top \mathbf{V}_m \mathbf{V}_m^\top \mathbf{q} = \mathbf{q}^\top \mathbf{P}\mathbf{q} = \mathbf{q}^\top \mathbf{P}^\top \mathbf{P}\mathbf{q} = \|\mathbf{P}\mathbf{q}\|_2^2.$$

(c) Finally, show that

$$\|\mathbf{x}_i - \mathbf{x}_j\|_2^2 = \|\mathbf{y}_i - \mathbf{y}_j\|_2^2 + \|\mathbf{q} - \mathbf{P}\mathbf{q}\|_2^2.$$

Conclude that $\|\mathbf{y}_i - \mathbf{y}_j\|_2 \leq \|\mathbf{x}_i - \mathbf{x}_j\|_2$. When does equality hold? (2 points)

Solution. The vectors $\mathbf{P}\mathbf{q}$ and $\mathbf{q} - \mathbf{P}\mathbf{q}$ are orthogonal, so Pythagoras gives

$$\|\mathbf{q}\|_2^2 = \|\mathbf{P}\mathbf{q}\|_2^2 + \|\mathbf{q} - \mathbf{P}\mathbf{q}\|_2^2.$$

Parts (a) and (b) identify $\|\mathbf{P}\mathbf{q}\|_2^2$ with $\|\mathbf{y}_i - \mathbf{y}_j\|_2^2$, proving the identity. The remaining term is nonnegative, proving the inequality. Equality holds exactly when $\mathbf{q} = \mathbf{P}\mathbf{q}$, equivalently when $\mathbf{x}_i - \mathbf{x}_j \in S$.

(d) Abhi claims that classical MDS always maps distinct data points to distinct locations. Is this true? Explain how two distinct points could end up at the same location in the embedding. (1 point)

Hint: Think about $\mathbf{P}\mathbf{q}$. When is this zero? What does that imply about $y_i - y_j$?

Solution. The claim is false. If $\mathbf{x}_i - \mathbf{x}_j$ is a nonzero vector perpendicular to S, then $\mathbf{P}(\mathbf{x}_i - \mathbf{x}_j) = \mathbf{0}$. Parts (a) and (b) then give $\|\mathbf{y}_i - \mathbf{y}_j\|_2^2 = 0$, so $\mathbf{y}_i = \mathbf{y}_j$ even though $\mathbf{x}_i \neq \mathbf{x}_j$.