Quiz 09

November 12th, 2026 15 minutes | 5 points

Name _____

Instructions. Answer all three parts. Show your work and briefly justify each answer.

1. Keeping an embedding spread out (Homework 08, Problem 3) 5 points

Let G be a graph on $n \geq 2$ vertices with symmetric weights $w_{ij} = w_{ji} \geq 0$, no self-loops ($w_{ii} = 0$), and no isolated vertices. Define the degree of vertex i, the degree matrix, and the total degree by

$$d_i = \sum_{j=1}^{n} w_{ij} > 0, \qquad \mathbf{D} = \operatorname{diag}(d_1, \ldots, d_n) \in \mathbb{R}^{n \times n}, \qquad V = \sum_{i=1}^{n} d_i > 0.$$

For an integer $1 \leq m < n$, an embedding is a matrix $\mathbf{Y} \in \mathbb{R}^{n \times m}$ whose ith row is $\mathbf{y}_i^\top$, where $\mathbf{y}_i \in \mathbb{R}^m$ is the point assigned to vertex i. Write $\mathbf{1}_n \in \mathbb{R}^n$ for the all-ones column vector, $\mathbf{0}_m \in \mathbb{R}^m$ for the zero vector, and $\mathbf{I}_m \in \mathbb{R}^{m \times m}$ for the identity matrix. The two embedding constraints are

$$\underbrace{\mathbf{Y}^\top \mathbf{D} \mathbf{1}_n = \mathbf{0}_m}_{\text{centering}}, \qquad \underbrace{\mathbf{Y}^\top \mathbf{D} \mathbf{Y} = \mathbf{I}_m}_{\text{normalization}}.$$

Finally, the embedding energy is

$$E(\mathbf{Y}) = \frac{1}{2} \sum_{i,j=1}^{n} w_{ij} \|\mathbf{y}_i - \mathbf{y}_j\|_2^2.$$

You may use the result from the homework without proof: every embedding satisfying both constraints obeys

$$\frac{1}{2} \sum_{i,j=1}^{n} d_i d_j \|\mathbf{y}_i - \mathbf{y}_j\|_2^2 = mV.$$

(a) Use the given identity to explain, *briefly*, how minimizing $E(\mathbf{Y})$ subject to both constraints can pull neighbours together without collapsing the entire embedding to a single point. (2 points)

Solution. The constraints fix the total degree-weighted spread over all pairs at $mV > 0$, while the energy penalizes distances between neighbours. Neighbours can move closer provided other pairwise distances increase to keep the total spread fixed. Complete collapse would make every distance zero, contradicting the identity.

(b) Suppose that Abhi dubiously claims that normalization alone is enough to prevent the entire embedding from collapsing to a single point. For $m = 1$, consider the constant embedding

$$\mathbf{Y} = \frac{1}{\sqrt{V}}\mathbf{1}_n \in \mathbb{R}^{n\times 1}, \qquad y_i = \frac{1}{\sqrt{V}} \quad \text{for every } i.$$

Evaluate $\mathbf{Y}^\top \mathbf{D}\mathbf{1}_n$ and $\mathbf{Y}^\top \mathbf{D}\mathbf{Y}$, and use your results to prove him wrong. (2 points)

Solution. It violates centering, since

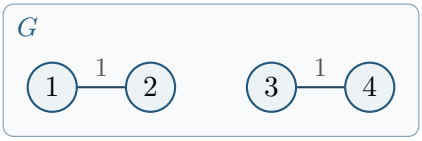
$$\mathbf{Y}^\top \mathbf{D}\mathbf{1}_n = \frac{1}{\sqrt{V}} \sum_{i=1}^{n} d_i = \sqrt{V} \neq 0.$$

It satisfies normalization, since

$$\mathbf{Y}^\top \mathbf{D}\mathbf{Y} = \frac{1}{V} \sum_{i=1}^{n} d_i = 1 = \mathbf{I}_1.$$

All vertices occupy the same point, so this is a normalized but completely collapsed embedding. Abhi's claim is false.

(c) Now consider the graph G shown at right, with edge weights labelled. Give a one-dimensional embedding $\mathbf{Y} \in \mathbb{R}^{4\times 1}$ satisfying centering, normalization, and $E(\mathbf{Y}) = 0$. (1 point)



Solution. Here $\mathbf{D} = \mathbf{I}_4$ and $V = 4$. One choice is

$$\mathbf{Y} = \frac{1}{2}(1, 1, -1, -1)^\top.$$

It satisfies both constraints, since

$$\mathbf{Y}^\top \mathbf{D}\mathbf{1}_4 = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} - \frac{1}{2} = 0, \qquad \mathbf{Y}^\top \mathbf{D}\mathbf{Y} = 4 \cdot \frac{1}{4} = 1.$$

Every edge joins vertices at the same location, so

$$E(\mathbf{Y}) = (y_1 - y_2)^2 + (y_3 - y_4)^2 = 0.$$

Each component collapses to its own point, but the two points are distinct. The spread comes from the four unordered pairs in different components, each at distance 1, so the total degree-weighted spread is $4 = mV$ as required. These pairs contribute nothing to the energy because there are no edges between the components.