Quiz 08

November 5th, 2026 15 minutes | 5 points

Name _____

Instructions. Answer all three parts. Show your work and briefly justify each answer. Parts (a) and (b) are worth 2 points each and part (c) is worth 1 point. Use the space provided.

1. Adding an edge never hurts (Homework 07, Problem 4) 5 points

Let G be a weighted graph on $n \geq 2$ vertices with symmetric weights $w_{k\ell} \geq 0$, no self-loops ($w_{kk} = 0$), and Laplacian $\mathbf{L} = \mathbf{D} - \mathbf{W}$ with eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis vectors of $\mathbb{R}^n$ and $\mathbf{1} \in \mathbb{R}^n$ the all-ones vector. You may use the following two facts without proof.

Fact 1 (Homework 07, Problem 3).

$$\mathbf{L} = \sum_{k<\ell} w_{k\ell}(\mathbf{e}_k - \mathbf{e}_\ell)(\mathbf{e}_k - \mathbf{e}_\ell)^\top.$$

Fact 2 (Rayleigh–Ritz). For the Laplacian of any weighted graph on n vertices, in particular for both $\mathbf{L}$ and $\mathbf{L}'$ below,

$$\lambda_2 = \min\{\mathbf{z}^\top \mathbf{L}\mathbf{z} : \ \|\mathbf{z}\|_2 = 1, \ \mathbf{z} \perp \mathbf{1}\},$$

and the minimum is attained.

Fix a pair $i \neq j$ and $w > 0$. Let G' be G with the weight of $\{i, j\}$ increased by w (if $\{i, j\}$ was not an edge, this adds it), with Laplacian $\mathbf{L}'$ and second eigenvalue λ_2'.

(a) Show that $\mathbf{L}' = \mathbf{L} + w(\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_i - \mathbf{e}_j)^\top$. Conclude that $\lambda_2' \geq \lambda_2$: adding an edge never makes a graph less connected. (2 points)

Solution. In Fact 1 for G', every term is the same as for G except the one for $\{i, j\}$, whose weight is $w_{ij} + w$ instead of w_{ij}. So $\mathbf{L}' - \mathbf{L} = w(\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_i - \mathbf{e}_j)^\top$. For every $\mathbf{z} \in \mathbb{R}^n$, $\mathbf{z}^\top(\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_i - \mathbf{e}_j)^\top\mathbf{z} = ((\mathbf{e}_i - \mathbf{e}_j)^\top\mathbf{z})^2 = (z_i - z_j)^2$, so

$$\mathbf{z}^\top \mathbf{L}'\mathbf{z} = \mathbf{z}^\top \mathbf{L}\mathbf{z} + w(z_i - z_j)^2 \geq \mathbf{z}^\top \mathbf{L}\mathbf{z}.$$

By Fact 2 for $\mathbf{L}'$, pick a unit $\mathbf{z} \perp \mathbf{1}$ with $\mathbf{z}^\top \mathbf{L}'\mathbf{z} = \lambda_2'$. This $\mathbf{z}$ is also a candidate in the minimum defining λ_2, so

$$\lambda_2' = \mathbf{z}^\top \mathbf{L}'\mathbf{z} \geq \mathbf{z}^\top \mathbf{L}\mathbf{z} \geq \lambda_2.$$

(b) Let $\mathbf{u}_2$ be a unit eigenvector of $\mathbf{L}$ for λ_2 with $\mathbf{u}_2 \perp \mathbf{1}$, and write $u_2(k)$ for its k-th entry. Show that $\lambda_2' \le \lambda_2 + w\big(u_2(i) - u_2(j)\big)^2$. (2 points)

Solution. Since $\mathbf{u}_2$ is a unit vector orthogonal to $\mathbf{1}$, it is a candidate in the minimum defining λ_2' in Fact 2. Since $\mathbf{L}\mathbf{u}_2 = \lambda_2\mathbf{u}_2$ and $\|\mathbf{u}_2\|_2 = 1$, we have $\mathbf{u}_2^\top \mathbf{L}\mathbf{u}_2 = \lambda_2$. So by the identity from part (a),

$$\lambda_2' \le \mathbf{u}_2^\top \mathbf{L}'\mathbf{u}_2 = \mathbf{u}_2^\top \mathbf{L}\mathbf{u}_2 + w\big(u_2(i) - u_2(j)\big)^2 = \lambda_2 + w\big(u_2(i) - u_2(j)\big)^2.$$

(c) Now suppose n is even and G has exactly two connected components C_1 and C_2, each with $n/2$ vertices. Then $\lambda_2 = 0$, and you may use without proof that $\mathbf{u}_2 = \frac{1}{\sqrt{n}}(\mathbf{1}_{C_1} - \mathbf{1}_{C_2})$ is a unit eigenvector of $\mathbf{L}$ for λ_2 with $\mathbf{u}_2 \perp \mathbf{1}$. Use parts (a) and (b) to show the following. (1 point)

- If $i, j \in C_1$ (an edge inside a component), then $\lambda_2' = 0$.
- If $i \in C_1$ and $j \in C_2$ (an edge between the components), then $\lambda_2' \le 4w/n$.

Solution. If $i, j \in C_1$, then $u_2(i) = u_2(j) = 1/\sqrt{n}$, so part (b) gives $\lambda_2' \le 0$, while part (a) gives $\lambda_2' \ge \lambda_2 = 0$. Hence $\lambda_2' = 0$, as it should be: G' is still disconnected.
If $i \in C_1$ and $j \in C_2$, then $u_2(i) - u_2(j) = 2/\sqrt{n}$, so part (b) gives $\lambda_2' \le 0 + w \cdot 4/n = 4w/n$. Only an edge across the gap between the components can raise λ_2 at all.