

Quiz 05

October 8th, 2026

15 minutes | 5 points

Name _____

Instructions. Answer both parts. Show your work and briefly justify each answer. Part (a) is worth 2 points and part (b) is worth 3 points. Use the space provided.

1. Least squares with unequal noise (Homework 04, Problem 4)

5 points

The design matrix $\mathbf{X} \in \mathbb{R}^{n \times d}$ is fixed, and $\mathbf{y} = \mathbf{X}\boldsymbol{\theta} + \boldsymbol{\varepsilon}$ for a fixed unknown $\boldsymbol{\theta} \in \mathbb{R}^d$. Recall the standard assumptions:

- (A1) **Zero-mean noise:** $\mathbb{E}[\boldsymbol{\varepsilon}] = \mathbf{0}$.
- (A2) **Homoscedastic, uncorrelated noise:** $\text{Cov}(\boldsymbol{\varepsilon}) = \sigma^2 \mathbf{I}_n$ for some $\sigma^2 > 0$.
- (A3) **Full column rank:** $\text{rank}(\mathbf{X}) = d$.

Under (A1)–(A3), the least squares estimator $\hat{\boldsymbol{\theta}}_{\text{LS}} = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{y}$ satisfies $\mathbb{E}[\hat{\boldsymbol{\theta}}_{\text{LS}}] = \boldsymbol{\theta}$ and $\text{Cov}(\hat{\boldsymbol{\theta}}_{\text{LS}}) = \sigma^2 (\mathbf{X}^\top \mathbf{X})^{-1}$.

In this problem, keep (A1) and (A3), but replace (A2) by

- (A2') **General noise covariance:** $\text{Cov}(\boldsymbol{\varepsilon}) = \sigma^2 \mathbf{W}$ for some $\sigma^2 > 0$ and some fixed positive semidefinite $\mathbf{W} \in \mathbb{R}^{n \times n}$.

You may use that for a random vector $\mathbf{z} \in \mathbb{R}^m$ and fixed $\mathbf{A} \in \mathbb{R}^{k \times m}$, $\mathbf{b} \in \mathbb{R}^k$,

$$\mathbb{E}[\mathbf{A}\mathbf{z} + \mathbf{b}] = \mathbf{A}\mathbb{E}[\mathbf{z}] + \mathbf{b}, \quad \text{Cov}(\mathbf{A}\mathbf{z} + \mathbf{b}) = \mathbf{A} \text{Cov}(\mathbf{z}) \mathbf{A}^\top.$$

(a) Show that $\hat{\boldsymbol{\theta}}_{\text{LS}}$ is still unbiased, that is, $\mathbb{E}[\hat{\boldsymbol{\theta}}_{\text{LS}}] = \boldsymbol{\theta}$. (2 points)

Solution. Write $\hat{\boldsymbol{\theta}}_{\text{LS}} = \mathbf{A}\mathbf{y}$ with the fixed matrix $\mathbf{A} = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top$. By (A1), $\mathbb{E}[\mathbf{y}] = \mathbf{X}\boldsymbol{\theta} + \mathbb{E}[\boldsymbol{\varepsilon}] = \mathbf{X}\boldsymbol{\theta}$, so

$$\mathbb{E}[\hat{\boldsymbol{\theta}}_{\text{LS}}] = \mathbf{A} \mathbb{E}[\mathbf{y}] = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{X} \boldsymbol{\theta} = \boldsymbol{\theta}.$$

Only (A1) and (A3) were used, so changing (A2) does not affect unbiasedness.

(b) Show that

$$\text{Cov}(\hat{\boldsymbol{\theta}}_{\text{LS}}) = \sigma^2 (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{W} \mathbf{X} (\mathbf{X}^\top \mathbf{X})^{-1},$$

and check that this agrees with the formula above when $\mathbf{W} = \mathbf{I}_n$. (3 points)

Solution. Since $\mathbf{X}\boldsymbol{\theta}$ is fixed, $\text{Cov}(\mathbf{y}) = \text{Cov}(\mathbf{X}\boldsymbol{\theta} + \boldsymbol{\varepsilon}) = \text{Cov}(\boldsymbol{\varepsilon}) = \sigma^2 \mathbf{W}$. With $\mathbf{A} = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top$ as in part (a), and using that $(\mathbf{X}^\top \mathbf{X})^{-1}$ is symmetric, $\mathbf{A}^\top = \mathbf{X}(\mathbf{X}^\top \mathbf{X})^{-1}$. Therefore

$$\text{Cov}(\hat{\boldsymbol{\theta}}_{\text{LS}}) = \mathbf{A}(\sigma^2 \mathbf{W})\mathbf{A}^\top = \sigma^2 (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{W} \mathbf{X} (\mathbf{X}^\top \mathbf{X})^{-1}.$$

When $\mathbf{W} = \mathbf{I}_n$, the middle factor is $\mathbf{X}^\top \mathbf{X}$, which cancels one $(\mathbf{X}^\top \mathbf{X})^{-1}$:

$$\text{Cov}(\hat{\boldsymbol{\theta}}_{\text{LS}}) = \sigma^2 (\mathbf{X}^\top \mathbf{X})^{-1} (\mathbf{X}^\top \mathbf{X}) (\mathbf{X}^\top \mathbf{X})^{-1} = \sigma^2 (\mathbf{X}^\top \mathbf{X})^{-1}.$$