

## Quiz 03

September 24th, 2026

15 minutes | 5 points

Name \_\_\_\_\_

**Instructions.** Answer both questions. Show your work and briefly justify each answer. Each part is worth 1 point. Use the space provided.

### 1. Orthogonal matrices (Homework 02, Problem 3)

3 points

Let  $\mathbf{Q} \in \mathbb{R}^{m \times m}$  be orthogonal, so  $\mathbf{Q}^\top \mathbf{Q} = \mathbf{I}_m$ , and let  $\mathbf{A} \in \mathbb{R}^{m \times n}$ .

- (a) Show that  $\|\mathbf{Q}\mathbf{x}\|_2 = \|\mathbf{x}\|_2$  for every  $\mathbf{x} \in \mathbb{R}^m$ .

**Solution.** Using  $\mathbf{Q}^\top \mathbf{Q} = \mathbf{I}_m$ ,

$$\|\mathbf{Q}\mathbf{x}\|_2^2 = (\mathbf{Q}\mathbf{x})^\top (\mathbf{Q}\mathbf{x}) = \mathbf{x}^\top \mathbf{Q}^\top \mathbf{Q} \mathbf{x} = \mathbf{x}^\top \mathbf{x} = \|\mathbf{x}\|_2^2.$$

Taking square roots gives  $\|\mathbf{Q}\mathbf{x}\|_2 = \|\mathbf{x}\|_2$ .

- (b) What are the singular values of  $\mathbf{Q}$ ?

**Solution.** The singular values of  $\mathbf{Q}$  are the square roots of the eigenvalues of  $\mathbf{Q}^\top \mathbf{Q} = \mathbf{I}_m$ . Every eigenvalue of  $\mathbf{I}_m$  is 1, so  $\mathbf{Q}$  has  $m$  singular values, all equal to  $\sqrt{1} = 1$ . An orthogonal matrix stretches nothing.

- (c) Show that  $\mathbf{QA}$  has the same singular values as  $\mathbf{A}$ .

**Solution.** We compute

$$(\mathbf{QA})^\top (\mathbf{QA}) = \mathbf{A}^\top \mathbf{Q}^\top \mathbf{QA} = \mathbf{A}^\top \mathbf{A}.$$

The singular values of  $\mathbf{QA}$  and  $\mathbf{A}$  are the square roots of the eigenvalues of the same matrix, so they are equal.

**2. Follow-up: the converse****2 points**

- (a) Let  $\mathbf{A} \in \mathbb{R}^{n \times n}$  be a square matrix whose  $n$  singular values are all equal to 1. Prove that  $\mathbf{A}$  is orthogonal, that is,  $\mathbf{A}^\top \mathbf{A} = \mathbf{I}_n$ .

**Solution.** Write the SVD  $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$  with  $\mathbf{U}, \mathbf{V} \in \mathbb{R}^{n \times n}$  orthogonal. Every singular value is 1, so  $\mathbf{\Sigma} = \mathbf{I}_n$  and  $\mathbf{A} = \mathbf{U}\mathbf{V}^\top$ . Then

$$\mathbf{A}^\top \mathbf{A} = \mathbf{V}\mathbf{U}^\top \mathbf{U}\mathbf{V}^\top = \mathbf{V}\mathbf{V}^\top = \mathbf{I}_n.$$

(Equivalently:  $\mathbf{A}^\top \mathbf{A}$  is symmetric with every eigenvalue equal to  $\sigma_i^2 = 1$ , so its eigendecomposition is  $\mathbf{A}^\top \mathbf{A} = \mathbf{V}\mathbf{I}_n\mathbf{V}^\top = \mathbf{I}_n$ .) Squareness matters:  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  has singular value 1 but is not orthogonal.